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Differential reductions of the Kadomtsev–Petviashvili equation and associated higher dimensional nonlinear PDEs

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Abstract

We represent an algorithm allowing one to construct new classes of partially integrable multidimensional nonlinear partial differential equations (PDEs) starting with the special type of solutions to the (1 + 1)-dimensional hierarchy of nonlinear PDEs linearizable by the matrix Hopf–Cole substitution (the Bürgers hierarchy). We derive examples of four-dimensional nonlinear matrix PDEs together with the scalar and three-dimensional reductions. Variants of the Kadomtsev–Petviashvili-type and Korteweg–de Vries-type equations are represented among them. Our algorithm is based on the combination of two Frobenius-type reductions and special differential reduction imposed on the matrix fields of integrable PDEs. It is shown that the derived four-dimensional nonlinear PDEs admit arbitrary functions of two variables in their solution spaces which clarifies the integrability degree of these PDEs.

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1. Introduction

The problem of the construction of such multidimensional nonlinear partial differential equations (PDEs) which are either completely integrable or, at least, possess a big manifold of particular solutions is a very attractive problem of integrability theory. First of all, one should emphasize several remarkable classical works regarding the completely integrable models such as [1] (where the Korteweg–de Vries equation (KdV) has been first time studied), [2, 3] (where the dressing method for a big class of soliton and instanton equations has been formulated) and [4] (where the so-called Sato approach to the integrability is applied to the Kadomtsev–Petviashvili equation (KP)). A big class of the first-order systems of quasilinear PDEs is integrated in [5, 6] using the generalized hodograph method. However, most of the above integrable nonlinear PDEs are (2+1)- and/or (1+1)-dimensional, excepting the selfdual-type PDEs [7, 8] and the equations associated with commuting vector fields [9–14].

In this paper we use the modification of the algorithm represented in [15, 16] allowing one to construct new multidimensional partially integrable nonlinear PDEs. It is shown in [16] that the Frobenious reduction of the matrix fields of the nonlinear PDEs integrable either by the Hoph–Cole substitution [17] (C-integrable PDE [18–23]) or by the method of characteristics [24, 25] (Ch-integrable PDE) leads to one of two big classes of the nonlinear PDEs integrable by the inverse spectral transform method (ISTM) [26–29] (S-integrable PDEs [18]). These classes are (a) soliton equations, such as KdV [1, 30], the nonlinear Schrödinger equation (NLS) [31], the Kadomtsev–Petviashvili equation (KP) [32], the Deavi–Stewartson equation (DS) [33], and (b) instanton equations, such as the self-dual Yang–Mills equation (SDYM) [7, 8].

The natural question is whether the Frobenius reduction (or its modification) can be used for the construction of new types of integrable (or at least partially integrable) systems starting with any known integrable system or this method works only for the derivation of soliton and instanton PDEs from C- and Ch-integrable ones?

At first glance the answer is negative. In fact, one can verify that Frobenius reduction imposed on the matrix field of S-integrable PDE (such as the GL(N) SDYM, the N-wave equation and the KP) produces the same S-integrable PDE for the block of the original matrix field. However, there is a method to generate new higher dimensional partially integrable systems of nonlinear PDEs using the Frobenius-type reduction after the appropriate differential reduction imposed on the matrix fields of the above S-integrable nonlinear PDEs.

Such a combination of reductions has already been used in [34]. It is shown there that GL(N) SDYM supplemented by the pair of reductions, namely the differential reduction relating certain blocks of the matrix field and the Frobenius reduction of these blocks, produces a new five-dimensional system of matrix nonlinear PDEs (with three-dimensional solution space) whose scalar reduction results in the nonlinear PDE associated with commuting vector fields [13, 14].

Following the strategy of [34], we consider an algorithm for the construction of new partially integrable PDEs starting with the matrix KP (although this algorithm may be applied to any S-integrable model). We will derive two representatives of matrix systems whose scalar reductions yield the following four-dimensional equations:

$$\left(v_{x\tau} - \tfrac{1}{2}v_{yt_2} - \tfrac{1}{2}\nu_0 v_{xx}\right)_y - \tfrac{1}{2}v_{xxx} + v_{yyy}v_x - v_y v_{xyy} = 0, \tag{1}$$

$$v_{xt_3} - \tfrac{3}{4}v_{t_2t_2} - \tfrac{1}{4}v_{xxxx} + \tfrac{3}{2}\left(v_{yt_2}v_x - v_{t_2}v_{xy} - \left(v_x\partial_y^{-1}v_{xx}\right)_x\right) = 0. \tag{2}$$

Another reduction,

$$v_{t_2} = 0, \tag{3}$$

reduces these PDEs into the following three-dimensional ones:

$$\left(v_{x\tau} - \tfrac{1}{2}\nu_0 v_{xx}\right)_y - \tfrac{1}{2}v_{xxx} + v_{yyy}v_x - v_y v_{xyy} = 0, \tag{4}$$

$$v_{t_3} - \tfrac{1}{4}v_{xxx} - \tfrac{3}{2}v_x\partial_y^{-1}v_{xx} = 0. \tag{5}$$

Here we take τ and t_3 as evolutionary parameters (times) while x, y and t_2 are taken as space parameters. Since the linear parts of equations (2) and (5) coincide with the linear parts of KP and KdV, respectively, equations (2) and (5) may be treated as new variants of KP- and

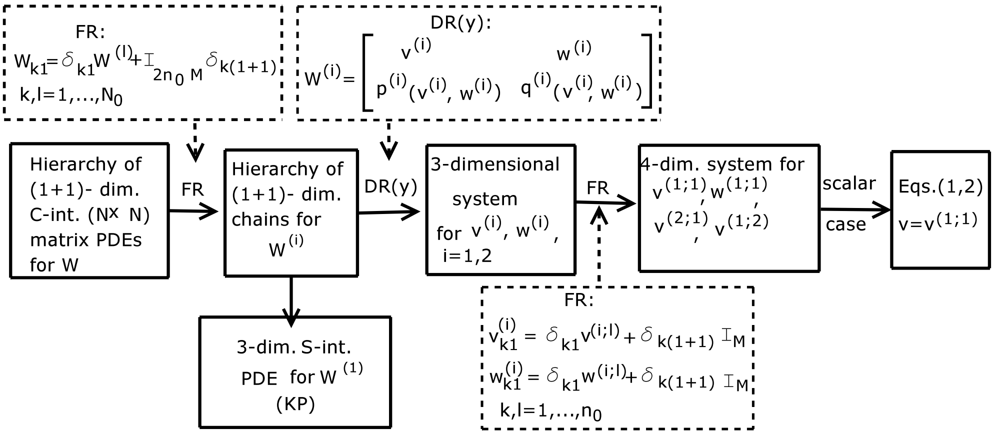

Figure 1. The chain of transformations from the (1+1)-dimensional C-integrable Bürgers hierarchy to the four-dimensional PDEs (1) and (2). Here $N = 2N_0n_0M$, $W^{(i)}$ are $2n_0M \times 2n_0M$ matrix fields, $v^{(i)}$, $w^{(i)}$, $p^{(i)}$ and $q^{(i)}$ are $n_0M \times n_0M$ blocks, I_{2n_0M} and I_M are $2n_0M$- and M-dimensional identity matrices, respectively. Blocks $p^{(i)}$ and $q^{(i)}$ are defined by equations (26). Here N_0, n_0 and M are arbitrary positive integer parameters.

KdV-type equations, respectively. The feature of these equations is that the derivative with respect to y appears only in the nonlinear parts. For this reason, these equations may be referred to as dispersionless ones.

Remember that KP originates from the (1+1)-dimensional C-integrable Bürgers hierarchy due to the Frobenius reduction [16]. Thus, the complete set of transformations leading to equations (1) and (2) is following (see also figure 1). We start with the C-integrable Bürgers hierarchy of nonlinear PDEs with independent variables x and t_n, $n = 2, 3, \ldots$. The Frobenius reduction of this hierarchy [16] yields the proper hierarchy of the discrete chains of nonlinear PDEs, which is equivalent to the chains obtained in the Sato approach to the integrability of (2+1)-dimensional KP [4]. The latter may be derived as an intermediate result of our algorithm after eliminating all extra fields using the combination of the first and the second representatives of the constructed discrete hierarchy. Next, apply the differential reduction introducing one more independent variable y. This step yields a three-dimensional system of nonlinear PDEs, i.e. dimensionality coincides with that of KP. Finally, the Frobenius-type reduction applied to the matrix fields of the latter nonlinear PDEs results in a four-dimensional system of matrix PDEs whose scalar versions yield equations (1), (2). It will be shown that the derived four-dimensional nonlinear PDEs may not be completely integrated by our method because the available solution spaces to them are restricted.

This paper is organized as follows. In section 2 we briefly recall some results of [16] and derive the discrete chains of nonlinear PDEs produced by the (1+1)-dimensional Bürgers hierarchy (with independent variables x and t_n, $n = 2, 3$) supplemented by the Frobenius reduction. Using a few equations of these chains we eliminate all extra fields and derive the matrix KP. In section 3 we suggest the special type differential reduction imposed on the blocks of the $2n_0M \times 2n_0M$ matrix fields of the above chains. This reduction introduces a new independent variables y and τ and allows one to generate $n_0M \times n_0M$ three-dimensional matrix PDEs. The Frobenius-type reduction of the above PDEs results in the four-dimensional systems of $M \times M$ matrix PDEs, section 4. Scalar versions of these PDEs result in equations (1) and (2) which, in particular, may be reduced to equations (4) and (5), respectively.

Solution spaces to the nonlinear PDEs derived in sections 3 and 4 will be studied in section 5. The obstacles to the complete integrability of equations (1), (2), (4) and (5) as well as the obstacles to their integrability by the ISTM are briefly discussed in section 6. Conclusions are given in section 7.

2. Relation between the (1 + 1)-dimensional *C*-integrable Bürgers hierarchy and the matrix KP

2.1. The (1+1)-dimensional C-integrable Bürgers hierarchy

Here we use the Bürgers hierarchy as a simplest example of *C*-integrable hierarchies linearizable by the matrix Hopf–Cole substitution. Namely, let W be the solution of the following linear algebraic matrix equation:

$$\hat\chi_x = W\hat\chi, \tag{6}$$

$$D_{t_n}\hat\chi = \partial_x^n\hat\chi, \qquad D_{t_n}\hat\chi \equiv \hat\chi_{t_n} + \hat\chi\Lambda^{(n)}, \qquad n = 2, 3, \ldots. \tag{7}$$

Here $\hat\chi$ and W are $2N_0n_0M \times 2N_0n_0M$ matrix functions, and $\Lambda^{(n)}$ are $2N_0n_0M \times 2N_0n_0M$ commuting constant matrices. To anticipate, the matrices $\Lambda^{(n)}$ are introduced in order to establish the reductions eliminating derivatives with respect to t_n from the nonlinear PDEs, see, for example, reduction (3). The compatibility conditions of equations (6) and (7) yield the linearizable Bürgers hierarchy. We will need only the first and the second representatives of this hierarchy below, i.e. $n = 2, 3$ in equations (7):

$$W_{t_2} - W_{xx} - 2W_xW = 0, \tag{8}$$

$$W_{t_3} - W_{xxx} - 3W_{xx}W - 3W_x(W_x + W^2) = 0. \tag{9}$$

2.2. The Frobenius reduction and associated chains of nonlinear PDEs

Introduce the Frobenius reduction [16]:

$$W = \begin{pmatrix} W^{(1)} & W^{(2)} & \cdots & W^{(N_0-1)} & W^{(N_0)} \\ I_{2Mn_0} & 0_{2Mn_0} & \cdots & 0_{2Mn_0} & 0_{2Mn_0} \\ 0_{2Mn_0} & I_{2Mn_0} & \cdots & 0_{2Mn_0} & 0_{2Mn_0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2Mn_0} & 0_{2Mn_0} & \cdots & I_{2Mn_0} & 0_{2Mn_0} \end{pmatrix}, \tag{10}$$

where I_J and 0_J are $J \times J$ identity and zero matrices, respectively, and $W^{(i)}$ are $2n_0M \times 2n_0M$ matrix functions. Substituting equation (10) into equation (6) we obtain the following block structure of $\hat\chi$:

$$\hat\chi = \begin{pmatrix} \chi^{(1)} & \cdots & \chi^{(N_0)} \\ \partial_x^{-1}\chi^{(1)} & \cdots & \partial_x^{-1}\chi^{(N_0)} \\ \cdots & \cdots & \cdots \\ \partial_x^{-N_0+1}\chi^{(1)} & \cdots & \partial_x^{-N_0+1}\chi^{(N_0)} \end{pmatrix}, \tag{11}$$

where $\chi^{(i)}$ are $2n_0M \times 2n_0M$ matrix functions. Thus, equations (6) and (7) may be written as follows:

$$\chi_x^{(m)}(\vec x) = \sum_{i=1}^{N_0} W^{(i)}(\vec x)\partial_x^{-i+1}\chi^{(m)}(\vec x), \tag{12}$$

$$\tilde{D}_{t_n}\chi^{(m)} = \partial_x^n \chi^{(m)}, \qquad \chi = (\chi^{(1)} \quad \cdots \quad \chi^{(N_0)}), \tag{13}$$

$$\begin{aligned} &\tilde{D}_{t_n}\chi^{(m)} = \chi^{(m)}_{t_n} + \chi^{(m)}\Lambda^{(n;m)}, \qquad \Lambda^{(n)} = \mathrm{diag}(\Lambda^{(n;1)}, \ldots, \Lambda^{(n;N_0)}),\\ &m = 1, \ldots, N_0, \quad n = 2, 3, \ldots, \end{aligned} \tag{14}$$

where $\Lambda^{(n;m)}$ are some $2n_0M \times 2n_0M$ commuting constant matrices. The meaning of the integer parameters n_0 and M is clarified in figure 1. The compatibility conditions of equations (12) and (13) (with $n = 2, 3$) yield the following discrete chains:

$$W^{(n)}_{t_2} - W^{(n)}_{xx} - 2W^{(n+1)}_x - 2W^{(1)}_x W^{(n)} = 0, \tag{15}$$

$$\begin{aligned} &W^{(n)}_{t_3} - W^{(n)}_{xxx} - 3\big(W^{(1)}_{xx} W^{(n)} + W^{(n+1)}_{xx} + W^{(1)}_x\big(W^{(1)}W^{(n)} + W^{(n+1)} + W^{(n)}_x\big)\\ &\qquad + W^{(2)}_x W^{(n)} + W^{(n+2)}_x\big) = 0, \qquad n = 1, \ldots, N_0,\\ &W^{(l)} = 0, \qquad l > N_0. \end{aligned} \tag{16}$$

Alternatively, these chains may be derived by substituting equation (10) into equations (8) and (9).

2.3. Matrix KP and its scalar reduction

The matrix KP is represented by the system of three equations involving fields $W^{(i)}, i = 1, 2, 3$: equation (15) with $n = 1, 2$ and equation (16) with $n = 1$. Eliminating $W^{(2)}$ and $W^{(3)}$ from this system one gets the following nonlinear PDE for the field $W^{(1)}$:

$$\big(W^{(1)}_{t_3} - \tfrac{1}{4}W^{(1)}_{xxx} - \tfrac{3}{2}\big(W^{(1)}_x\big)^2\big)_x + \tfrac{3}{2}\big[W^{(1)}_x, W^{(1)}_{t_2}\big] - \tfrac{3}{4}W^{(1)}_{t_2t_2} = 0, \tag{17}$$

where square parenthesis means matrix commutator. In the scalar case this equation reduces to the following one, $u \equiv W^{(1)}$:

$$\big(u_{t_3} - \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}u_x^2\big)_x - \tfrac{3}{4}u_{t_2t_2} = 0 \tag{18}$$

which is the scalar potential KP.

3. Differential reduction of the matrix KP

Let matrices $\chi^{(j)}$ have the following block structure:

$$\chi^{(j)} = \begin{pmatrix} \Psi^{(2j-1)} & \Psi^{(2j)} \\ \Psi^{(2j-1)}_y & \Psi^{(2j)}_y \end{pmatrix}, \qquad j = 1, \ldots, N_0, \tag{19}$$

where $\Psi^{(m)}$ are $n_0M \times n_0M$ matrix functions. We introduce the y-dependence of $\Psi^{(m)}$ by the following second-order PDE:

$$\mathcal{E}^{(0)} := \Psi^{(m)}_{yy} = a\Psi^{(m)}_x + \nu\Psi^{(m)}_y + \mu\Psi^{(m)}, \qquad m = 1, \ldots, 2N_0. \tag{20}$$

Here a, ν and μ are $n_0M \times n_0M$ diagonal constant matrices. The block structure of $\chi^{(j)}$ (19) suggests us the relevant block structure of $W^{(j)}$:

$$\begin{aligned} &W^{(j)} = \begin{pmatrix} w^{(j)} & v^{(j)} \\ p^{(j)} & q^{(j)} \end{pmatrix}, \qquad j = 1, \ldots, N_0,\\ &v^{(l)} = w^{(l)} = p^{(l)} = q^{(l)} = 0_{Mn_0}, \qquad l > N_0, \end{aligned} \tag{21}$$

where $w^{(j)}, v^{(j)}, q^{(j)}$ and $p^{(j)}$ are $n_0M \times n_0M$ matrix functions. Now matrix $2n_0M \times 2n_0M$ equations (12) may be written as two $n_0M \times 2N_0n_0M$ equations:

$$\mathcal{E}^{(1)} := \Psi_x = \sum_{i=1}^{N_0} \left(v^{(i)} \partial_x^{-i+1} \Psi_y + w^{(i)} \partial_x^{-i+1} \Psi \right), \tag{22}$$

$$\mathcal{E}^{(2)} := \Psi_{xy} = \sum_{i=1}^{N_0} \left(q^{(i)} \partial_x^{-i+1} \Psi_y + p^{(i)} \partial_x^{-i+1} \Psi \right), \tag{23}$$

where

$$\Psi = (\Psi^{(1)}, \dots, \Psi^{(2N_0)}). \tag{24}$$

The compatibility condition of equations (22) and (23),

$$\mathcal{E}_y^{(1)} = \mathcal{E}^{(2)}, \tag{25}$$

yields the expressions for $p^{(j)}$ and $q^{(j)}$ in terms of $v^{(j)}$ and $w^{(j)}$:

$$p^{(j)} = w_y^{(j)} + v^{(j)}\mu + v^{(j+1)}a + v^{(1)}aw^{(j)}, \qquad q^{(j)} = v_y^{(j)} + v^{(j)}\nu + w^{(j)} + v^{(1)}av^{(j)}. \tag{26}$$

Thus, only two blocks of $W^{(j)}$ are independent, i.e. $w^{(j)}$ and $v^{(j)}$. Matrix equation (22) may be considered as the uniquely solvable system of $2N_0$ linear $n_0M \times n_0M$ matrix algebraic equations for the matrix functions $v^{(i)}$ and $w^{(i)}$, $i = 1, \dots, N_0$, while equation (23) is the consequence of equation (22).

Equations (13) yield

$$\begin{aligned} &\hat{D}_{t_n}\Psi^{(m)} = \partial_x^n \Psi^{(m)}, \qquad \hat{D}_{t_n}\Psi^{(m)} \equiv \Psi_{t_n}^{(m)} + \Psi^{(m)}\tilde{\Lambda}^{(n;m)}, \qquad m = 1, \dots, 2N_0, \\ &\Lambda^{(n;j)} = \mathrm{diag}(\tilde{\Lambda}^{(n;2j-1)}, \tilde{\Lambda}^{(n;2j)}), \qquad j = 1, \dots, N_0, \qquad n = 2, 3, \dots, \end{aligned} \tag{27}$$

where $\tilde{\Lambda}^{(n,m)}$ are $n_0M \times n_0M$ commuting constant matrices. Equation (20) allows us to introduce one more set of parameters τ_n as follows:

$$\Psi_{\tau_n}^{(m)} = \partial_x^n \Psi_y^{(m)}, \qquad m = 1, \dots, 2N_0, \qquad n = 1, 2, \dots. \tag{28}$$

Before proceeding further we select the following three equations out of the system (27), (28):

$$\hat{D}_{t_2}\Psi^{(m)} = \Psi_{xx}^{(m)}, \tag{29}$$

$$\Psi_\tau^{(m)} \equiv \Psi_{\tau_1}^{(m)} = \Psi_{xy}^{(m)}, \tag{30}$$

$$\hat{D}_{t_3}\Psi^{(m)} = \Psi_{xxx}^{(m)}. \tag{31}$$

We also assume that a, ν and μ are scalars, i.e.

$$a = I_{n_0M}, \qquad \nu = \nu_0 I_{n_0M}, \qquad \mu = \mu_0 I_{n_0M}. \tag{32}$$

In order to derive the nonlinear PDEs for $v^{(i)}$ and $w^{(i)}$, we must consider the conditions providing the compatibility of equation (22) and equations (29)–(31):

$$\begin{aligned} &\hat{D}_{t_2}\mathcal{E}^{(1)} - \mathcal{E}_{xx}^{(1)} = 0 \quad \Rightarrow \\ &\sum_{i=1}^{N_0} E_1^{(1i)} \partial_x^{-i+1} D_y \Psi + \sum_{i=1}^{N_0} E_0^{(1i)} \partial_x^{-i+1} \Psi = 0, \qquad n = 1, 2, \dots, \end{aligned} \tag{33}$$

$$\begin{aligned} &\mathcal{E}_\tau^{(1)} - \mathcal{E}_{xy}^{(1)} = 0 \quad \Rightarrow \\ &\sum_{i=1}^{N_0} E_1^{(2i)} \partial_x^{-i+1} \Psi_y + \sum_{i=1}^{N_0} E_0^{(2i)} \partial_x^{-i+1} \Psi = 0, \qquad n = 1, 2, \dots, \end{aligned} \tag{34}$$

$$\hat{D}_{t_3}\mathcal{E}^{(1)} - \mathcal{E}^{(1)}_{xxx} = 0 \quad \Rightarrow$$
$$\sum_{i=1}^{N_0} E_1^{(3i)}\partial_x^{-i+1}\Psi_y + \sum_{i=1}^{N_0} E_0^{(3i)}\partial_x^{-i+1}\Psi = 0, \qquad n = 1, 2, \ldots. \tag{35}$$

These equations generate the following chains of nonlinear PDEs for $v^{(i)}$ and $w^{(i)}$, $i = 1, \ldots, N_0$:

$$E_1^{(1i)} \equiv v^{(i)}_{t_2} - v^{(i)}_{xx} - 2v^{(i+1)}_x - 2\big(\nu_0 v^{(1)}_x + w^{(1)}_x + v^{(1)}_x v^{(1)}\big)v^{(i)} - 2v^{(1)}_x\big(w^{(i)} + v^{(i)}_y\big) = 0, \tag{36}$$

$$E_0^{(1i)} \equiv w^{(i)}_{t_2} - w^{(i)}_{xx} - 2w^{(i+1)}_x - 2\big(w^{(1)}_x + v^{(1)}_x v^{(1)}\big)w^{(i)} - 2v^{(1)}_x\big(w^{(i)}_y + v^{(i+1)} + \mu_0 v^{(i)}\big) = 0, \tag{37}$$

$$E_1^{(2i)} \equiv v^{(i)}_\tau - v^{(i)}_{xy} - v^{(i+1)}_y - \nu_0 v^{(i)}_x - w^{(i)}_x - \big(\nu_0 v^{(1)}_y + w^{(1)}_y + v^{(1)}_x + v^{(1)}_y v^{(1)}\big)v^{(i)}$$
$$- v^{(1)}_y\big(w^{(i)} + v^{(i)}_y\big) = 0, \tag{38}$$

$$E_0^{(2i)} \equiv w^{(i)}_\tau - w^{(i)}_{xy} - w^{(i+1)}_y - v^{(i+1)}_x - \mu_0 v^{(i)}_x - \big(w^{(1)}_y + v^{(1)}_x + v^{(1)}_y v^{(1)}\big)w^{(i)}$$
$$- v^{(1)}_y\big(v^{(i+1)} + w^{(i)}_y + \mu_0 v^{(i)}\big) = 0, \tag{39}$$

$$E_1^{(3i)} \equiv v^{(i)}_{t_3} - v^{(i)}_{xxx} - 3v^{(i+1)}_{xx} - 3v^{(i+2)}_x - 3\big(w^{(2)}_x + w^{(1)}_{xx} + v^{(1)}_x v^{(2)} + v^{(1)}_x w^{(1)}_y + \big(v^{(1)}_x\big)^2$$
$$+ v^{(2)}_x v^{(1)} + w^{(1)}_x w^{(1)} + v^{(1)}_{xx} v^{(1)} + v^{(1)}_x v^{(1)} w^{(1)} + v^{(1)}_x w^{(1)} v^{(1)} + v^{(1)}_x v^{(1)}_y v^{(1)} + w^{(1)}_x (v^{(1)})^2$$
$$+ v^{(1)}_x (v^{(1)})^3 + \nu_0\big(v^{(1)}_{xx} + v^{(1)}_x w^{(1)} + v^{(1)}_x v^{(1)}_y + v^{(2)}_x + w^{(1)}_x v^{(1)} + \nu_0 v^{(1)}_x v^{(1)} + 2v^{(1)}_x (v^{(1)})^2\big)$$
$$+ \mu_0 v^{(1)}_x v^{(1)}\big)v^{(i)} - 3v^{(1)}_x\big(w^{(i+1)} + v^{(i+1)}_y + w^{(i)}_x + v^{(i)}_{xy} + v^{(1)} v^{(i+1)} + v^{(1)} v^{(i)}_x$$
$$+ w^{(1)} v^{(i)}_y + v^{(1)}_y w^{(i)} + v^{(1)}_y v^{(i)}_y + (v^{(1)})^2 w^{(i)} + (v^{(1)})^2 v^{(i)}_y + w^{(1)} w^{(i)} + \nu_0\big(v^{(i+1)} + v^{(i)}_x$$
$$+ v^{(1)} w^{(i)} + v^{(1)} v^{(i)}_y\big)\big) - 3\big(v^{(2)}_x\big(w^{(i)} + v^{(i)}_y\big) + w^{(1)}_x v^{(i)}_x + w^{(1)}_x v^{(i+1)} + v^{(1)}_{xx} v^{(i)}_y + v^{(1)}_{xx} w^{(i)}$$
$$+ w^{(1)}_x\big(v^{(1)} w^{(i)} + v^{(1)} v^{(i)}_y\big)\big) = 0, \tag{40}$$

$$E_0^{(3i)} \equiv w^{(i)}_{t_3} - w^{(i)}_{xxx} - 3w^{(i+1)}_{xx} - 3w^{(i+2)}_x - 3\big(w^{(1)}_{xx} + w^{(2)}_x + v^{(2)}_x v^{(1)} + w^{(1)}_x w^{(1)} + v^{(1)}_{xx} v^{(1)}$$
$$+ v^{(1)}_x (v^{(1)} w^{(1)} + w^{(1)} v^{(1)} + v^{(1)}_y v^{(1)} + (v^{(1)})^3 + v^{(2)} + w^{(1)}_y + v^{(1)}_x) + w^{(1)}_x (v^{(1)})^2$$
$$+ \nu_0 v^{(1)}_x (v^{(1)})^2 + \mu_0 v^{(1)}_x v^{(1)}\big)w^{(i)} - 3v^{(1)}_x\big(v^{(i+2)} + w^{(i+1)}_y + v^{(i+1)}_x + w^{(i)}_{xy} + v^{(1)} w^{(i+1)}$$
$$+ v^{(1)} w^{(i)}_x + w^{(1)} v^{(i+1)} + w^{(1)} w^{(i)}_y + v^{(1)}_y v^{(i+1)} + v^{(1)}_y w^{(i)}_y + (v^{(1)})^2\big(v^{(i+1)} + w^{(i)}_y\big)$$
$$+ \nu_0\big(v^{(1)} v^{(i+1)} + v^{(1)} w^{(i)}_y\big) + \mu_0\big(v^{(1)}_y v^{(i)} + w^{(1)} v^{(i)} + v^{(i)}_x + (v^{(1)})^2 v^{(i)} + v^{(i+1)}$$
$$+ \nu_0 v^{(1)} v^{(i)}\big)\big) - 3v^{(2)}_x\big(v^{(i+1)} + w^{(i)}_y + \mu_0 v^{(i)}\big) - 3w^{(1)}_x\big(w^{(i+1)} + w^{(i)}_x + v^{(1)}(v^{(i+1)}$$
$$+ w^{(i)}_y + \mu_0 v^{(i)})\big) - 3v^{(1)}_{xx}\big(v^{(i+1)} + w^{(i)}_y + \mu_0 v^{(i)}\big). \tag{41}$$

In addition, we must take into account the compatibility condition of equations (20) and (22):

$$\mathcal{E}^{(1)}_{yy} = \mathcal{E}^{(1)}_x + \nu_0 \mathcal{E}^{(1)}_y + \mu_0 \mathcal{E}^{(1)} \quad \Rightarrow \quad \sum_{i=0}^{N_0}\big(\tilde{E}_1^{(i)}\partial_x^{-i+1}\Psi_y + \tilde{E}_0^{(i)}\partial_x^{-i+1}\Psi\big) = 0, \tag{42}$$

which gives us the following system of non-evolutionary nonlinear chains:

$$\tilde{E}_1^{(i)} := v^{(i)}_{yy} + 2w^{(i)}_y - v^{(i)}_x + \nu_0 v^{(i)}_y + 2v^{(1)}_y v^{(i)} = 0, \tag{43}$$

$$\tilde{E}_0^{(i)} := w_{yy}^{(i)} - \nu_0 w_y^{(i)} - w_x^{(i)} + 2v_y^{(i+1)} + 2\mu_0 v_y^{(i)} + 2v_y^{(1)} w^{(i)} = 0, \qquad i = 1, \dots, N_0. \tag{44}$$

One can show that the derived chains (36)–(40), (43), (44) generate three compatible three-dimensional systems of nonlinear PDEs as follows:

$$E_i^{(11)} = 0, \qquad \tilde{E}_i^{(1)} = 0, \qquad i = 0, 1, \qquad \text{fields } v^{(j)}, w^{(j)}, \quad j = 1, 2,$$
$$\text{independent variables } t_2, x, y, \tag{45}$$

$$E_i^{(21)} = 0, \qquad \tilde{E}_i^{(1)} = 0, \qquad i = 0, 1, \quad \text{fields } v^{(j)}, w^{(j)}, \quad j = 1, 2,$$
$$\text{independent variables } \tau, x, y, \tag{46}$$

$$E_i^{(31)} = 0, \qquad \tilde{E}_i^{(k)} = 0, \qquad i = 0, 1, \quad k = 1, 2, \quad \text{fields } v^{(j)}, w^{(j)}, \quad j = 1, 2, 3,$$
$$\text{independent variables } t_3, x, y. \tag{47}$$

We do not represent these equations explicitly since they are intermediate equations in our algorithm.

4. The Frobenius-type reduction, new four-dimensional systems of matrix nonlinear PDEs and their scalar reductions

Three systems (45)–(47) represent three commuting flows with times t_2, τ and t_3, respectively. In this section we follow the strategy of [16] and show that the hierarchy of nonlinear chains (36)–(41), (43), (44) supplemented by the Frobenius-type reduction of the matrix fields $v^{(i)}$ and $w^{(i)}$ generates the hierarchy of four-dimensional systems of nonlinear PDEs, see equations (65)–(68) and the text thereafter.

This is possible due to the remarkable property of chains of nonlinear PDEs (36)–(41), (43), (44). Namely, these chains admit the following Frobenious-type reduction:

$$\begin{aligned}
&v^{(i)} = \{v^{(i;kl)}, k, l = 1, \dots, n_0\}, \qquad w^{(i)} = \{w^{(i;kl)}, k, l = 1, \dots, n_0\},\\
&v^{(i;kl)} = \delta_{k1} v^{(i;l)} + \delta_{k(l+n_1(i))} I_M,\\
&w^{(i;kl)} = \delta_{k1} w^{(i;l)} + \delta_{k(l+n_2(i))} I_M, \qquad i = 1, \dots, N_0,\\
&v^{(i;l)} = w^{(i;l)} = 0_M, \qquad l > n_0, \quad \forall\, i,
\end{aligned} \tag{48}$$

where n_0 is an arbitrary positive integer parameter, $n_1(i)$ and $n_2(i)$ are arbitrary positive integer functions of positive integer argument and $v^{(i;l)}$ and $w^{(i;l)}$ are $M \times M$ matrix fields. In particular, if $n_1(i) = n_2(i) = 1$, then this reduction becomes Frobenius one [16], which is shown in figure 1.

Equations (48) require the following block structures of the functions $\Psi^{(l)}$ and of the constant matrices $\tilde{\Lambda}^{(i;l)}$:

$$\begin{aligned}
&\Psi^{(l)} = \{\Psi^{(l;nm)}, \ n, m = 1, \dots, n_0\},\\
&\tilde{\Lambda}^{(i;l)} = \{\Lambda^{(i;l;m)}, \ m = 1, \dots, n_0\}, \qquad l = 1, \dots, 2N_0, \quad i = 2, 3.
\end{aligned} \tag{49}$$

Here $\Psi^{(l;nm)}$ are $M \times M$ matrix functions and $\Lambda^{(i;l;m)}$ are $M \times M$ commuting constant matrices. In turn, equation (22) reduces to the following one:

$$\Psi_x^{(l;nm)} = \sum_{i=1}^{N_0} \sum_{j=1}^{n_0} \left[\left(\delta_{n1} v^{(i;j)} + \delta_{n(j+n_1(i))}\right) \partial_x^{-i+1} \Psi_y^{(l;jm)} + \left(\delta_{n1} w^{(i;j)} + \delta_{n(j+n_2(i))}\right) \partial_x^{-i+1} \Psi^{(l;jm)} \right],$$
$$l = 1, \dots, 2N_0, \qquad n, m = 1, \dots, n_0, \tag{50}$$

while equations (29)–(31), (20) yield

$$\mathcal{D}_{t_2}\Psi^{(l;nm)} = \Psi^{(l;nm)}_{xx}, \tag{51}$$

$$\Psi^{(l;nm)}_{\tau} = \Psi^{(l;nm)}_{xy}, \tag{52}$$

$$\mathcal{D}_{t_3}\Psi^{(l;nm)} = \Psi^{(l;nm),}_{xxx} \tag{53}$$

$$\begin{aligned} &\Psi^{(l;nm)}_{yy} = \Psi^{(l;nm)}_{x} + \nu_0\Psi^{(l;nm)}_{y} + \mu_0\Psi^{(l;nm)},\\ &\mathcal{D}_{t_i}\Psi^{(l;nm)} = \Psi^{(l;nm)}_{t_i} + \Psi^{(l;nm)}\Lambda^{(i;l;m)}, \qquad i = 2, 3,\\ &l = 1, \ldots, 2N_0, \qquad n, m = 1, \ldots, n_0. \end{aligned} \tag{54}$$

Then the chains of nonlinear PDEs (36)–(41), (43), (44) get the following block structures:

$$\begin{aligned} &E^{(ni)}_m = \left\{E^{(ni;l)}_m\delta_{k1}, k, l, = 1, \ldots, n_0\right\} = 0, \qquad n = 1, 2, 3,\\ &\tilde{E}^{(i)}_m = \left\{\tilde{E}^{(i;l)}_m\delta_{k1}, k, l, = 1, \ldots, n_0\right\} = 0,\\ &m = 0, 1, \qquad i = 1, \ldots, N_0. \end{aligned} \tag{55}$$

In particular, nonlinear chains (36), (37) and (43), (44) with the Frobenious-type reduction (48) generate the following chains with two discrete indexes:

$$\begin{aligned} E^{(1i;l)}_1 \equiv{}& v^{(i;l)}_{t_2} - v^{(i;l)}_{xx} - 2v^{(i+1;l)}_x - 2\big(\nu_0 v^{(1;1)}_x + w^{(1;1)}_x + v^{(1;1)}_x v^{(1;1)}\big)v^{(i;l)} - 2v^{(1;1)}_x\big(w^{(i;l)}\\ &+ v^{(i;l)}_y\big) - 2\big(\nu_0 v^{(1;l+n_1(i))}_x + w^{(1;l+n_1(i))}_x + v^{(1;1+n_1(1))}_x v^{(i;l)} + v^{(1;1)}_x v^{(1;l+n_1(i))}\\ &+ v^{(1;l+n_1(1)+n_1(i))}_x + v^{(1;l+n_2(i))}_x\big) = 0, \end{aligned} \tag{56}$$

$$\begin{aligned} E^{(1i;l)}_0 \equiv{}& w^{(i;l)}_{t_2} - w^{(i;l)}_{xx} - 2w^{(i+1;l)}_x - 2\big(w^{(1;1)}_x + v^{(1;1)}_x v^{(1;1)}\big)w^{(i;l)}\\ &- 2v^{(1;1)}_x\big(w^{(i;l)}_y + v^{(i+1;l)} + \mu_0 v^{(i;l)}\big)\\ &- 2\big(w^{(1;l+n_1(i))}_x + v^{(1;1+n_1(1))}_x w^{(i;l)} + v^{(1;1)}_x v^{(1;l+n_2(i))}\\ &+ v^{(1;l+n_1(1)+n_2(i))}_x + v^{(1;l+n_1(i+1))}_x + \mu_0 v^{(1;l+n_1(i))}_x\big) = 0, \end{aligned} \tag{57}$$

$$\tilde{E}^{(i;l)}_1 := v^{(i;l)}_{yy} + 2w^{(i;l)}_y - v^{(i;l)}_x + \nu_0 v^{(i;l)}_y + 2v^{(1;1)}_y v^{(i;l)} + 2v^{(i;l+n_1(i))}_y = 0, \tag{58}$$

$$\tilde{E}^{(i;l)}_0 := w^{(i;l)}_{yy} - \nu_0 w^{(i;l)}_y - w^{(i;l)}_x + 2v^{(i+1;l)}_y + 2\mu_0 v^{(i;l)}_y + 2v^{(1;1)}_y w^{(i;l)} + 2v^{(i;l+n_2(i))}_y = 0, \tag{59}$$

where $i = 1, \ldots, N_0, l = 1, \ldots, n_0$. Now our goal is to write the complete system of PDEs for some fields $v^{(i;j)}$ and/or $w^{(n;m)}$ which would be independent on both parameters N_0 and n_0 (in the spirit of the Sato theory [4]). This may be done if, along with the system (56)–(59), one involves the discrete chains of PDEs generated by either equations (38), (39) or equations (40), (41) together with the Frobenious-type reduction (48). For instance, the system (38), (39) yields

$$\begin{aligned} E^{(2i;l)}_1 \equiv{}& v^{(i;l)}_{\tau} - v^{(i;l)}_{xy} - v^{(i+1;l)}_y - \nu_0 v^{(i;l)}_x - w^{(i;l)}_x - \big(\nu_0 v^{(1;1)}_y + w^{(1;1)}_y + v^{(1;1)}_x + v^{(1;1)}_y v^{(1;1)}\big)v^{(i;l)}\\ &- v^{(1;1)}_y\big(w^{(i;l)} + v^{(i;l)}_y\big) - \big(\nu_0 v^{(i;l+n_1(i))}_y + w^{(i;l+n_1(i))}_y + v^{(i;l+n_1(i))}_x\\ &+ v^{(1;1+n_1(1))}_y v^{(i;l)} + v^{(1;1)}_y v^{(1;l+n_1(i))} + v^{(1;l+n_1(1)+n_1(i))}_y\big) - v^{(1;l+n_2(i))}_y = 0, \end{aligned} \tag{60}$$

$$\begin{aligned} E^{(2i;l)}_0 \equiv{}& w^{(i;l)}_{\tau} - w^{(i;l)}_{xy} - w^{(i+1;l)}_y - v^{(i+1;l)}_x - \mu_0 v^{(i;l)}_x - \big(w^{(1;1)}_y + v^{(1;1)}_x + v^{(1;1)}_y v^{(1;1)}\big)w^{(i;l)}\\ &- v^{(1;1)}_y\big(v^{(i+1;l)} + w^{(i;l)}_y + \mu_0 v^{(i;l)}\big) - w^{(1;l+n_2(i))}_y - v^{(1;l+n_2(i))}_x - \big(v^{(1;1+n_1(1))}_y w^{(i;l)}\\ &+ v^{(1;1)}_y v^{(1;l+n_2(i))} + v^{(1;l+n_1(1)+n_2(i))}_y\big) - v^{(1;l+n_1(i+1))}_y - v^{(1;l+n_1(i))}_y\mu_0 = 0. \end{aligned} \tag{61}$$

The discrete chains generated by equations (40) and (41) are very cumbersome. However, it will be shown that we need only equation (40) with $i = 1$ for the derivation of the complete system of nonlinear PDEs. The latter equation may be simplified after the elimination of $v_x^{(2)}$, $v_x^{(3)}$ and $w_x^{(2)}$ using equation (36) with $i = 1, 2$ and equation (37) with $i = 1$. This simplification is reasonable because, in any case, we are going to use chain (56) (produced by equations (36)) in order to derive the complete system of nonlinear PDEs. One gets in result:

$$4v_{t_3}^{(1)} - v_{xxx}^{(1)} - 3v_{xt_2}^{(1)} - 6v_{t_2}^{(2)} - 6\left(w_{t_2}^{(1)} + v_{t_2}^{(1)}v^{(1)} + \left(v_x^{(1)}\right)^2 + \nu_0 v_{t_2}^{(1)}\right)v^{(1)}$$
$$-6v_x^{(1)}\left(w_x^{(1)} + v_{xy}^{(1)} + v^{(1)}v_x^{(1)} + \nu_0 v_x^{(1)}\right) - 6v_{t_2}^{(1)}\left(w^{(1)} + v_y^{(1)}\right) - 6w_x^{(1)}v_x^{(1)} = 0. \quad (62)$$

After the Frobenius reduction, this equation generates the following discrete chain:

$$4v_{t_3}^{(1;l)} - v_{xxx}^{(1;l)} - 3v_{xt_2}^{(1;l)} - 6v_{t_2}^{(2;l)} - 6\left(w_{t_2}^{(1;1)} + v_{t_2}^{(1;1)}v^{(1;1)} + \left(v_x^{(1;1)}\right)^2 + \nu_0 v_{t_2}^{(1;1)}\right)v^{(1;l)}$$
$$- 6v_x^{(1;1)}\left(w_x^{(1;l)} + v_{xy}^{(1;l)} + v^{(1;1)}v_x^{(1;l)} + \nu_0 v_x^{(1;l)}\right) - 6v_{t_2}^{(1;1)}\left(w^{(1;l)} + v_y^{(1;l)}\right)$$
$$- 6w_x^{(1;1)}v_x^{(1;l)} - 6\left(w_{t_2}^{(1;l+n_1(1))} + v_{t_2}^{(1;l+2n_1(1))} + v_{t_2}^{(1;1+n_1(1))}v^{(1;l)} + v_{t_2}^{(1;1)}v^{(1;l+n_1(1))}\right.$$
$$\left.+ v_x^{(1;1)}v_x^{(1;l+n_1(1))} + \nu_0 v_{t_2}^{(1;l+n_1(1))} + v_x^{(1;1+n_1(1))}v_x^{(1;l)} + v_{t_2}^{(1;l+n_2(1))}\right) = 0. \quad (63)$$

Having chains of equations (56)–(61), (63) we would like to exhibit the complete system of nonlinear PDEs independent of both N_0 and n_0 taking a few equations out of these chains. After some examination of these chains we conclude that it is enough to take the equations $E_1^{(k1;1)}$, $k = 1, 2$, $\tilde{E}_1^{(31;1)}$ and $\tilde{E}_1^{(1;1)}$. Introducing the new fields

$$v = v^{(1;1)}, \qquad q = v^{(1;1+n_1(1))} + w^{(1;1)},$$
$$p = v^{(2;1)} + w^{(1;1+n_1(1))} + v^{(1;1+n_2(1))} + v^{(1;1+2n_1(1))} + \nu_0 v^{(1;1+n_1(1))}, \quad (64)$$

we write these equations as follows:

$$E_1^{(11;1)} \equiv v_{t_2} - v_{xx} - 2p_x - 2(\nu_0 v_x + q_x + v_x v)v - 2v_x(q + v_y) = 0, \quad (65)$$

$$E_1^{(21;1)} \equiv v_\tau - v_{xy} - p_y - \nu_0 v_x - q_x - (\nu_0 v_y + q_y + v_x + v_y v)v - v_y(q + v_y) = 0, \quad (66)$$

$$\tilde{E}_1^{(31;1)} := 4v_{t_3} - v_{xxx} - 3v_{xt_2} - 6p_{t_2} - 6\left(q_{t_2} + v_{t_2}v + (v_x)^2 + \nu_0 v_{t_2}\right)v$$
$$- 6v_x(q_x + v_{xy} + vv_x + \nu_0 v_x) - 6v_{t_2}(q + v_y) - 6q_x v_x = 0, \quad (67)$$

$$\tilde{E}_1^{(1;1)} := v_{yy} + 2q_y - v_x + \nu_0 v_y + 2v_y v = 0. \quad (68)$$

Any two equations out of the system (65)–(67) supplemented by equation (68) represent the complete system of matrix nonlinear PDEs for the fields v, p and q. Two examples of the scalar nonlinear PDEs are given below in section 4.1.

4.1. *Examples of scalar nonlinear PDEs*

Example 1. Consider the system (65), (66), (68). In the scalar case this system reduces to the following single scalar nonlinear PDE for the field v:

$$\left(\left(E_1^{(21;1)}\right)_x - \tfrac{1}{2}\left(E_1^{(11;1)}\right)_y\right)_y + \tfrac{1}{2}\left(\tilde{E}_1^{(1;1)}\right)_{xx} = 0 \quad \Rightarrow \quad \text{equation (1)}. \quad (69)$$

Example 2. Consider the system (65), (67), (68). In the scalar case this system reduces to the single equation for the field v as follows:

$$\left(\frac{1}{4}\tilde{E}_1^{(31;1)} + \frac{3v_x}{2}\partial_y^{-1}\left(\tilde{E}_1^{(1;1)}\right)_x\right)_x - \frac{3}{4}\left(E_1^{(11;1)}\right)_{t_2} \quad \Rightarrow \quad \text{equation (2)}. \quad (70)$$

5. Solutions to the derived nonlinear PDEs

5.1. Solutions to the nonlinear system (45)–(47)

Solutions to systems of matrix nonlinear PDEs (45)–(47) or, more general, to discrete chains (36)–(41), (43), (44) are obtainable in terms of the functions $\Psi^{(m)}$ $m = 1, \dots, 2N_0$ taken as solutions of the linear system (20), (29)–(31):

$$\Psi^{(l)}_{\alpha\beta}(\vec{x}) = \sum_{i=1}^{2} \int \mathrm{d}q\, \psi^{(l;i)}_{\alpha\beta}(q)\, \mathrm{e}^{qx+k^{(i)}y+(q^2-\tilde{\Lambda}^{(2;l)}_{\beta})t_2+(q^3-\tilde{\Lambda}^{(3;l)}_{\beta})t_3+k^{(i)}q\tau},$$
$$\alpha,\beta = 1,\dots,n_0M, \qquad l = 1,\dots,2N_0, \tag{71}$$

where $\psi^{(l)}_{\alpha\beta}(q)$ are the arbitrary scalar functions of single scalar variable, $\vec{x} = (x, y, t_2, t_3)$ is the list of all independent variables of the nonlinear PDEs and $k^{(i)}$, $i = 1, 2$, are roots of the characteristic equation associated with equation (20) where a, ν and μ are given by equations (32):

$$\begin{aligned} &k^2 - q - \nu_0 k - \mu_0 = 0, \quad \Rightarrow \\ &k^{(1)} = \tfrac{1}{2}\big(\nu_0 + \sqrt{\nu_0^2 + 4q + 4\mu_0}\big), \qquad k^{(2)} = \tfrac{1}{2}\big(\nu_0 - \sqrt{\nu_0^2 + 4q + 4\mu_0}\big). \end{aligned} \tag{72}$$

Now we can use equation (22) in order to find $v^{(i)}$ and $w^{(i)}$. In particular, there is a big class of solutions in the form of rational functions of exponents, such as solitary wave solutions. We do not represent their explicit form since equations (45)–(47) are intermediate equations in our method of solving equations (1), (2).

Note nevertheless that the simplest nontrivial solution to equations (45)–(47) corresponds to $N_0 = 2$. Then equation (22) reduces to the following four matrix equations:

$$\Psi^{(m)}_x = \sum_{i=1}^{4} \big(v^{(i)}\partial_x^{-i+1}\Psi^{(m)}_y + w^{(i)}\partial_x^{-i+1}\Psi^{(m)}\big), \qquad m = 1, 2, 3, 4. \tag{73}$$

These equations, in general, are uniquely solvable for the matrix fields $v^{(i)}$ and $w^{(i)}$, $i = 1, 2$.

5.1.1. The reductions to the lower dimensional PDEs.

The described solution space admits the following reductions:

$$v^{(i)}_{t_n} = w^{(i)}_{t_n} = 0, \qquad \forall\, i, \tag{74}$$

where n is either 2 or 3. In fact, these reductions mean

$$\psi^{(l;i)}_{\alpha\beta}(q) \sim \delta\big(q^n - \tilde{\Lambda}^{(n;l)}_{\beta}\big) \tag{75}$$

in equation (71). For instance, let $n = 2$. Then we must substitute the following expression into equation (71):

$$\psi^{(l;i)}_{\alpha\beta}(q) = \psi^{(l;i1)}_{\alpha\beta}\delta\Big(q - \sqrt{\tilde{\Lambda}^{(n;l)}_{\beta}}\Big) + \psi^{(l;i2)}_{\alpha\beta}\delta\Big(q + \sqrt{\tilde{\Lambda}^{(n;l)}_{\beta}}\Big), \tag{76}$$

where $\psi^{(l;ik)}_{\alpha\beta}$, $k = 1, 2$, are arbitrary constants.

5.2. Solutions to the system (65)–(68) and to scalar equations (1), (2) and (4), (5)

In this case the fields $v^{(i;j)}$ and $w^{(i;j)}$ are solutions to the linear algebraic system (50). More precisely, only part of the system (50) is needed to define $v^{(i;j)}$ and $w^{(i;j)}$. In fact, the system

(50) may be viewed as two subsystems. The first one corresponds to $n = 1$:

$$\Psi_x^{(l;1m)} = \sum_{i=1}^{N_0}\sum_{j=1}^{n_0}\left[v^{(i;j)}\partial_x^{-i+1}\Psi_y^{(l;jm)} + w^{(i;j)}\partial_x^{-i+1}\Psi^{(l;jm)}\right],$$
$$l = 1,\ldots,2N_0, \qquad m = 1,\ldots,n_0. \tag{77}$$

This is the system of $2N_0n_0$ linear algebraic $M \times M$ matrix equations for the same number of the matrix fields $v^{(i;l)}$ and $w^{(i;l)}$, $i = 1,\ldots,N_0$, $j = 1,\ldots,n_0$. Namely equations (77) yield the functions $v^{(1;1)}$, $w^{(1;1)}$, $v^{(2;1)}$, $v^{(1;1+n_1(1))}$, $w^{(1;1+n_1(1))}$, $v^{(1;1+n_2(1))}$, $v^{(1;1+2n_1(1))}$ defining the solutions v, p, q to the system (65)–(68) in accordance with formulae (64). The second subsystem corresponds to $n > 1$ in (50):

$$\Psi_x^{(l;nm)} = \sum_{i=0}^{N_0}\left[\partial_x^{-i+1}\Psi_y^{(l;(n-n_1(i))m)} + \partial_x^{-i+1}\Psi^{(l;(n-n_2(i))m)}\right], \qquad l = 1,\ldots,2N_0,$$
$$n, m = 1,\ldots,n_0, \qquad \Psi^{(l;ij)} = 0, \quad \text{if,} \quad i \leqslant 0. \tag{78}$$

This subsystem expresses recursively the functions $\Psi^{(l;nm)}$, $n > 1$, in terms of the functions $\Psi^{(l;1m)}$. The simplest case corresponds to $n_j(i) = 1, \forall i, j$ (the Frobenius reduction).

The functions $\Psi^{(l;nm)}$ are solutions to the system (51)–(54) and may be written as follows:

$$\Psi_{\alpha\beta}^{(l;1m)}(\vec{x}) = \sum_{i=1}^{2}\int \mathrm{d}q\, \psi_{\alpha\beta}^{(lm;i)}(q)\, \mathrm{e}^{qx+k^{(i)}y+(q^2-\Lambda_\beta^{(2;l;m)})t_2+(q^3-\Lambda_\beta^{(3;l;m)})t_3+k^{(i)}q\tau}$$
$$\alpha,\beta = 1,\ldots,M, \qquad m = 1,\ldots,n_0, \qquad l = 1,\ldots,2N_0, \tag{79}$$

$$k^{(1)} = \frac{1}{2}\left(\nu_0 + \sqrt{\nu_0^2 + 4q + 4\mu_0}\right), \qquad k^{(2)} = \frac{1}{2}\left(\nu_0 - \sqrt{\nu_0^2 + 4q + 4\mu_0}\right). \tag{80}$$

Here $\psi_{\alpha\beta}^{(lm;i)}(q)$ are arbitrary scalar functions of one scalar argument. In particular, the derived formulae describe the big class of solutions having the form of rational functions of exponents, such as solitary wave solutions. The simplest examples of them will be given below, see equations (88)–(93), (96).

In the scalar case (corresponding to equations (1) and (2)) one has $M = 1$ so that equation (79) reads

$$\Psi^{(l;1m)}(\vec{x}) = \sum_{i=1}^{2}\int \mathrm{d}q\, \psi^{(lm;i)}(q)\, \mathrm{e}^{qx+k^{(i)}y+(q^2-\Lambda^{(2;l;m)})t_2+(q^3-\Lambda^{(3;l;m)})t_3+k^{(i)}q\tau} \tag{81}$$

with the same expressions for $k^{(i)}$ given by equation (80).

5.2.1. The reductions to the lower dimensional PDEs. Note that the described solution space admits reduction (3) embedded into the following set of reductions:

$$v_{t_n}^{(i;j)} = w_{t_n}^{(i;j)} = 0, \qquad \forall\, i, j, \tag{82}$$

where n is either 2 or 3. In fact, similar to reduction (74), these reductions mean

$$\psi_{\alpha\beta}^{(lm;i)}(q) \sim \delta\left(q^n - \Lambda_\beta^{(n;l;m)}\right) \tag{83}$$

in equation (79). For instance, let $n = 2$ which corresponds to reduction (3). Then we must substitute the following expression into equation (79):

$$\psi_{\alpha\beta}^{(lm;i)}(q) = \psi_{\alpha\beta}^{(lm;i1)}\delta\left(q - \sqrt{\Lambda_\beta^{(n;l;m)}}\right) + \psi_{\alpha\beta}^{(lm;i2)}\delta\left(q + \sqrt{\Lambda_\beta^{(n;l;m)}}\right), \tag{84}$$

where $\psi^{(lm;ik)}_{\alpha\beta}$ $(k = 1, 2)$ are arbitrary constants. In the scalar case (corresponding to nonlinear equations (4) and (5)) equation (84) reads

$$\psi^{(lm;i)}(q) = \psi^{(lm;i1)}\delta\big(q - \sqrt{\Lambda^{(n;l;m)}}\big) + \psi^{(lm;i2)}\delta\big(q + \sqrt{\Lambda^{(n;l;m)}}\big), \tag{85}$$

which must be substituted into equation (81).

5.2.2. The simplest solution to equations (1) and (2). The simplest nontrivial solution to equations (1) and (2) corresponds to $N_0 = n_0 = 1$ and $\Lambda^{(n;l;m)} = 0$, $n = 2, 3$. Then equation (81) reads

$$\Psi^{(l;11)}(\vec{x}) = \sum_{i=1}^{2} \int \mathrm{d}q\, \psi^{(l1;i)}(q)\, \mathrm{e}^{qx+k^{(l1;i)}y+q^2t_2+q^3t_3+k^{(l1;i)}q\tau}, \qquad l = 1, 2. \tag{86}$$

Equation (77) gets the following form:

$$\Psi^{(l;11)}_x = v^{(1;1)}\Psi^{(l;11)}_y + w^{(1;1)}\Psi^{(l;11)}, \qquad l = 1, 2. \tag{87}$$

Their solution reads

$$v \equiv v^{(1;1)} = \frac{\Delta_1}{\Delta}, \qquad \Delta = \begin{vmatrix} \Psi^{(1;11)}_y & \Psi^{(1;11)} \\ \Psi^{(2;11)}_y & \Psi^{(2;11)} \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \Psi^{(1;11)}_x & \Psi^{(1;11)} \\ \Psi^{(2;11)}_x & \Psi^{(2;11)} \end{vmatrix}, \tag{88}$$

$$w^{(1;1)} = \frac{\Delta_2}{\Delta}, \qquad \Delta_2 = \begin{vmatrix} \Psi^{(1;11)}_y & \Psi^{(1;11)}_x \\ \Psi^{(2;11)}_y & \Psi^{(2;11)}_x \end{vmatrix}. \tag{89}$$

Formulae (88), (89) have four arbitrary functions of single variable $\psi^{(l1;i)}(q)$, $l, i = 1, 2$.

Let us write the explicit formulae for the particular case $\psi^{(l1;i)}(q) = \psi_{li}\delta(q - q_{li})$, where both ψ_{li} and q_{li} are arbitrary constants:

$$\Delta = \sum_{i,j=1}^{2} \mathrm{e}^{Q_{1i}+Q_{2j}}\big((-1)^{i+1}K_{1i} - (-1)^{j+1}K_{2j}\big)\psi^{(11;i)}\psi^{(21;j)}, \tag{90}$$

$$\Delta_1 = \sum_{i,j=1}^{2} \mathrm{e}^{Q_{1i}+Q_{2j}}(q_{1i} - q_{2j})\psi^{(11;i)}\psi^{(21;j)}, \tag{91}$$

$$Q_{li} = q_{li}x + k^{(l1;i)}y + q_{li}^2 t_2 + q_{li}^3 t_3 + k^{(l1;i)}q\tau, \tag{92}$$

$$k^{(l1;1)} = \frac{\nu_0}{2} + K_{l1}, \qquad k^{(l1;2)} = \frac{\nu_0}{2} - K_{l2}, \qquad K_{li} = \frac{1}{2}\sqrt{\nu_0^2 + 4q_{li} + 4\mu_0}, \quad i, l = 1, 2. \tag{93}$$

The solution v has no singularities if $\Delta \neq 0$, which is true if, for instance, the following relations are valid:

$$K_{1i} > K_{2i} > 0, \qquad i = 1, 2, \qquad \psi^{(11;2)} < 0, \qquad \psi^{(11;1)}, \psi^{(21;1)}, \psi^{(21;2)} > 0. \tag{94}$$

For example, let

$$K_{2i} = q_{2i} = 0, \qquad i = 1, 2, \quad \Rightarrow \quad \mu_0 = -\frac{\nu_0^2}{4}, \qquad \psi_{11} = \xi_1 > 0,$$
$$\psi_{12} = -\xi_2 < 0 \tag{95}$$

Then

$$v = \frac{\mathrm{e}^{Q_{11}}\xi_1 q_{11} - \mathrm{e}^{Q_{12}}\xi_2 q_{12}}{\mathrm{e}^{Q_{11}}q_{11}\xi_1 + \mathrm{e}^{Q_{12}}q_{12}\xi_2}, \tag{96}$$

which is the kink.

5.2.3. The simplest solution to equations (4) and (5). We take equation (81) with $N_0 = n_0 = m = 1$ and $\Lambda^{(3;l;m)} = 0$, $\Lambda^{2;l;m} \equiv \Lambda^{lm}$:

$$\Psi^{(l;11)}(\vec{x}) = \sum_{i=1}^{2} \int \mathrm{d}q\, \psi^{(l1;i)}(q)\, \mathrm{e}^{qx+k^{(l1;i)}y+(q^2-\Lambda^{(l1)})t_2+q^3t_3+k^{(l1;i)}q\tau}, \qquad l = 1, 2. \tag{97}$$

Note that equations (4) and (5) require reduction (3), which means that the function $\psi^{(l1;i)}(q)$ in equation (97) is given by equation (85). For instance, let

$$\psi^{(lm;ij)} = \delta_{ij}\psi_{lmi}. \tag{98}$$

Then one has expressions

$$\begin{aligned}
\Delta &= \sum_{i,j=1}^{2} \mathrm{e}^{Q_{1i}+Q_{2j}}\big((-1)^{i+1}K_{1i} - (-1)^{j+1}K_{2j}\big)\psi_{11i}\psi_{21j},\\
\Delta_1 &= \sum_{i,j=1}^{2} \mathrm{e}^{Q_{1i}+Q_{2j}}\big((-1)^{i+1}\sqrt{\Lambda^{(11)}} - (-1)^{j+1}\sqrt{\Lambda^{(21)}}\big)\psi_{11i}\psi_{21j},\\
Q_{li} &= q_{li}x + k^{(l1;i)}y + q_{li}^3t_3 + k^{(l1;i)}q_{li}\tau, \qquad q_{l1} = \sqrt{\Lambda^{(l1)}}, \qquad q_{l2} = -\sqrt{\Lambda^{(l1)}}
\end{aligned} \tag{99}$$

instead of expressions (90)–(92), while relations (93) remain the same. The conditions similar to (94) with replacement $\psi^{(ij;k)} \to \psi_{ijk}$ guarantee that the solution v is non-singular in this case as well.

Let, for example, $\psi_{212} = 0$, $\psi_{111} = \xi_1 > 0$, $\psi_{112} = -\xi_2 < 0$ and $K_{11} > K_{21} > 0$. Then one has

$$v = \frac{\mathrm{e}^{Q_{11}}\xi_1(q_{11} - q_{21}) + \mathrm{e}^{Q_{12}}\xi_2(q_{11} + q_{21})}{\mathrm{e}^{Q_{11}}(K_{11} - K_{21})\xi_1 + \mathrm{e}^{Q_{12}}(K_{11} + K_{21})\xi_2}, \tag{100}$$

which is the kink.

6. Obstacles to the complete integrability of equations (1) and (2)

It is important to note that our algorithm does not describe the full solution spaces to nonlinear PDEs (1), (2), (4) and (5). To justify this statement we consider equations (1) and (2), while equations (4) and (5) may be treated in a similar way. The simplest argument is following. By construction, equations (1) and (2) must be commuting flows, i.e. the expression

$$[\text{equation (1)}]_{t_3} - [\text{equation (2)}]_{y\tau} \tag{101}$$

must be identical to zero. However, the direct calculation shows that this expression yields the nonlocal three-dimensional PDE having rather complicated form (we do not represent it here). This means that the constructed solution space to equations (1) and (2) is two dimensional (in other words, it may depend on the arbitrary functions of two independent variables) while the full solution space to four-dimensional PDEs (1) and (2) must be three dimensional. The dimensionality of the solution space is confirmed in section 5 by formulae (81), which shows us that the solution space depends on $4N_0n_0$ arbitrary functions of single variable $\psi^{(lm;i)}(q)$, $l = 1, \ldots, 2N_0$, $m = 1, \ldots, n_0$, $i = 1, 2$. Since both N_0 and n_0 are arbitrary positive integers and may go to infinity, these functions may approximate arbitrary functions of two variables in the solution space. Thus, the possibility of having an arbitrary function of three independent variables in the solution space (which would provide the full integrability) remains an open problem for a further study. Without details, we remark that solution space to four-dimensional equation (1) (available due to our algorithm) is two dimensional as well, while solution spaces to three-dimensional equations (4) and (5) are one dimensional, which may be obtained owing to the formulae (81) and (85).

Since equations (1) and (2) have been derived from the matrix KP and the latter is a well-known (2+1)-dimensional PDE integrable by the ISTM, one can expect the same type of integrability of equations (1) and (2). In particular, one can expect that these equations are compatibility conditions of some overdetermined linear spectral problem derivable from the overdetermined linear spectral problem for the matrix KP. However, this is not true. In the next subsection we consider equation (2) as an example and show that the linear spectral problem for this equation is not well defined. Thus, at the moment, the ISTM is not a suitable tool for solving equation (2). The same conclusion is valid for equations (1), (4) and (5) as well.

6.1. The spectral problem for the matrix KP supplemented by differential reduction (21), (26) and Frobenious-type reduction (48)

Although the algorithm described above allows one to find a big manifold of solutions to equations (1), (2), (4) and (5), it does not give us an algorithm to derive the linear spectral problem for any of these equations, i.e. at the moment, we are not able to obtain such overdetermined system of linear equations for some spectral function whose compatibility condition results in equation (1), or equation (2), or equation (4), or equation (5) without additional requirements to the coefficients of a linear system. As an example explaining this problem, we consider equation (2) as nonlinear equation derivable from the matrix KP (17) after differential reduction (introduced in section 3) followed by the Frobenious-type reduction (introduced in section 4) in this subsection.

The obstacle to derive the linear spectral problem for equation (2) is associated with the spectral representation of differential reduction (21), (26) which is a reduction of the potentials of the linear spectral problem for the matrix KP. In other words, we do not know which reduction must be imposed on the spectral function to generate the above differential reduction for the potentials of the spectral problem. Thus, the problem of spectral representation of equation (2) remains open as well as the problem of its complete integrability. Nevertheless, we derive some linear spectral problem whose compatibility condition leads to equation (2) imposing the above differential reduction 'by hand'.

The spectral problem for the matrix KP (17) reads

$$\begin{aligned} F^{(1)} &:= \psi_{t_2}(\lambda;\vec{x}) + \psi_{xx}(\lambda;\vec{x}) + 2\psi(\lambda;\vec{x})W^{(1)}_x(\vec{x}) = 0,\\ F^{(2)} &:= \psi_{t_3}(\lambda;\vec{x}) - \psi_{xxx}(\lambda;\vec{x}) - 3\psi_x(\lambda;\vec{x})W^{(1)}_x(\vec{x}) + \tfrac{3}{2}\psi(\lambda;\vec{x})\big(W^{(1)}_{t_2}(\vec{x}) - W^{(1)}_{xx}(\vec{x})\big) = 0, \end{aligned} \tag{102}$$

where ψ is a spectral function and λ is a spectral parameter. Differential reduction (21), (26) requires the following block structure of ψ:

$$\psi = \begin{pmatrix} \psi^{(11)} & \psi^{(12)} \\ \psi^{(21)} & \psi^{(22)} \end{pmatrix}, \tag{103}$$

where $\psi^{(ij)}$ are $n_0M \times n_0M$ matrices. So that equations (102) acquire the following block structure:

$$F^{(i)} \equiv \begin{pmatrix} F^{(i;11)} & F^{(i;12)} \\ F^{(i;21)} & F^{(i;22)} \end{pmatrix} = 0, \qquad i = 1, 2, \tag{104}$$

where $F^{(i;nm)}$ are $n_0M \times n_0M$ matrix equations. The spectral problem is now represented by the first-row blocks of equation (104):

$$F^{(1;11)} := \psi^{(11)}_{t_2} + \psi^{(11)}_{xx} + 2\psi^{(11)} w^{(1)}_x + 2\psi^{(12)} p^{(1)}_x = 0, \tag{105}$$

$$F^{(1;12)} := \psi^{(12)}_{t_2} + \psi^{(12)}_{xx} + 2\psi^{(11)} v^{(1)}_x + 2\psi^{(12)} q^{(1)}_x = 0, \tag{106}$$

$$F^{(2;11)} := \psi^{(11)}_{t_3} - \psi^{(11)}_{xxx} - 3\big(\psi^{(11)}_x w^{(1)}_x + \psi^{(12)}_x p^{(1)}_x\big) + \tfrac{3}{2}\big(\psi^{(11)}\big(w^{(1)}_{t_2} - w^{(1)}_{xx}\big) + \psi^{(12)}\big(p^{(1)}_{t_2} - p^{(1)}_{xx}\big)\big) = 0, \tag{107}$$

$$F^{(2;12)} := \psi^{(12)}_{t_3} - \psi^{(12)}_{xxx} - 3\big(\psi^{(11)}_x v^{(1)}_x + \psi^{(12)}_x q^{(1)}_x\big) + \tfrac{3}{2}\big(\psi^{(11)}\big(v^{(1)}_{t_2} - v^{(1)}_{xx}\big) + \psi^{(12)}\big(q^{(1)}_{t_2} - q^{(1)}_{xx}\big)\big) = 0, \tag{108}$$

while the second-row blocks of equation (104) are equivalent to the system (105)–(108) up to the replacement of the spectral functions $\psi^{(1k)} \to \psi^{(2k)}$, $k = 1, 2$. Emphasize that the differential reduction is already imposed on the coefficients of the above linear system due to the special form of the functions $p^{(1)}$ and $q^{(1)}$ given by equation (26). It may be shown by the direct calculations that the compatibility conditions of equations (105)–(108) yield the system of nonlinear PDEs which is equivalent to the system (45), (47), i.e. we derive two complete compatible systems of nonlinear PDEs so that equations (45) describe the t_2-evolution while equations (47) describe the t_3-evolution of fields. However, since differential reduction has no spectral representation, these nonlinear systems may not be integrated by the ISTM. Of course, this means that equation (2) (which has been derived from the above system of nonlinear PDEs after the Frobenious-type reduction (48)) may not be integrated by the ISTM as well. However, for the sake of completeness, we consider the Frobenious-type reduction (48) imposed on the linear system (105)–(108).

First of all, note that this reduction requires the following structure of the functions $\psi^{(11)}$ and $\psi^{(12)}$:

$$\psi^{(1n)} = \begin{pmatrix} \psi^{(n;11)} & \cdots & \psi^{(n;1n_0)} \\ \cdots & \cdots & \cdots \\ \psi^{(n;n_0 1)} & \cdots & \psi^{(n;n_0 n_0)} \end{pmatrix}, \qquad n = 1, 2, \tag{109}$$

where $\psi^{(n;ij)}$ are $M \times M$ functions. Equations (105)–(108) acquire the following block structure:

$$F^{(n;1m)} = \begin{pmatrix} F^{(nm;11)} & \cdots & F^{(nm;1n_0)} \\ \cdots & \cdots & \cdots \\ F^{(nm;n_0 1)} & \cdots & F^{(nm;n_0 n_0)} \end{pmatrix}, \qquad n, m = 1, 2, \tag{110}$$

where $F^{(nm;ij)}$ are $M \times M$ matrix equations. Similar to equation (104), only the first-row blocks of these equations represent the system of independent spectral equations:

$$F^{(11;n)} := \psi^{(1;n)}_{t_2} + \psi^{(1;n)}_{xx} + 2\psi^{(1;1)} w^{(1;n)}_x + 2\psi^{(2;1)} p^{(1;1n)}_x + 2\psi^{(2;2)} p^{(1;2n)}_x = 0, \tag{111}$$

$$F^{(12;n)} := \psi^{(2;n)}_{t_2} + \psi^{(2;n)}_{xx} + 2\psi^{(1;1)} v^{(1;n)}_x + 2\psi^{(2;1)} q^{(1;1n)}_x + 2\psi^{(2;2)} q^{(1;2n)}_x = 0, \tag{112}$$

$$F^{(21;n)} := \psi^{(1;n)}_{t_3} - \psi^{(1;n)}_{xxx} - 3\left(\psi^{(1;1)}_x w^{(1;n)}_x + \psi^{(2;1)}_x p^{(1;1n)}_x + \psi^{(2;2)}_x p^{(1;2n)}_x\right) + \tfrac{3}{2}\big(\psi^{(1;1)}\big(w^{(1;n)}_{t_2} - w^{(1;n)}_{xx}\big) + \psi^{(2;1)}\big(p^{(1;1n)}_{t_2} - p^{(1;1n)}_{xx}\big) + \psi^{(2;2)}\big(p^{(1;2n)}_{t_2} - p^{(1;2n)}_{xx}\big)\big) = 0, \tag{113}$$

$$F^{(22;n)} := \psi^{(2;n)}_{t_3} - \psi^{(2;n)}_{xxx} - 3\big(\psi^{(1;1)}_x v^{(1;n)}_x + \psi^{(2;1)}_x q^{(1;1n)}_x + \psi^{(2;2)}_x q^{(1;2n)}_x\big) + \frac{3}{2}\big(\psi^{(1;1)}\big(v^{(1;n)}_{t_2} - v^{(1;n)}_{xx}\big) + \psi^{(2;1)}\big(q^{(1;1n)}_{t_2} - q^{(1;1n)}_{xx}\big) + \psi^{(2;2)}\big(q^{(1;2n)}_{t_2} - q^{(1;2n)}_{xx}\big)\big) = 0, \tag{114}$$

where $n = 1, \ldots, n_0$ and

$$p^{(1;1k)} = w_y^{(1;k)} + \mu_0 v^{(1;k)} + v^{(2;k)} + v^{(1;1)} w^{(1;k)} + v^{(1;k+n_2(1))}, \qquad p^{(1;2k)} = w^{(1;k)},$$
$$q^{(1;1k)} = v_y^{(1;k)} + \nu_0 v^{(1;k)} + w^{(1;k)} + v^{(1;1)} v^{(1;k)} + v^{(1;k+1)}, \qquad p^{(1;2k)} = v^{(1;k)}. \tag{115}$$

Here we take into account equation (32) and take $n_1(1) = 1$ without loss of generality. The system of linear PDEs (111)–(114), $n = 1, 2$, represents the overdetermined system whose compatibility condition results in the complete system of nonlinear PDEs which includes equations (65), (67) and (68) as a complete subsystem, whose scalar version ($M = 1$) results in equation (2).

7. Conclusions

We have constructed two examples of the four-dimensional nonlinear PDEs starting with the dressing method for the (1+1)-dimensional *C*-integrable Bürgers hierarchy and using the combination of the Frobenius type and the differential reductions. One of these examples, equation (2), has the same dispersion relation as KP does and may be referred to as the KP-type equation. As a consequence, its lower dimensional reduction, equation (5), has the same dispersion relation as KdV does and may be referred to as the KdV-type equation. Although the derived four-dimensional PDEs are not completely integrable by our technique (see section 6), we are able to supply a big solution manifold to them with solitary wave solutions as most simple examples. The new feature of this algorithm in comparison with one represented in [16] is the differential reduction which introduces the new independent variable y into the nonlinear PDEs. In turn, this variable allows one to introduce the set of new τ-variables by formula (28), which is a new method of increasing the dimensionality of solvable nonlinear PDEs. An important problem is to overcome the obstacle to the complete integrability of the derived four-dimensional nonlinear PDEs (1), (2) and their three-dimensional reductions (4), (5). It is also important to find the physical application of the derived nonlinear PDEs. Regarding this problem we must note that the additional independent variable y appears only in the nonlinear terms of equations (2) and (5), which usually means the existence of solutions with break of wave profiles, so that one can expect that these equations describe the break of wave profiles in the physical systems where KP and KdV appear.

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